# A Chebyshev Quadrature Rule for One Sided Finite Part Integrals

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This paper is concerned with a Chebyshev quadrature rule for approximating one sided finite part integrals with smooth density functions. Our quadrature rule is based on the Chebyshev interpolation polynomial with the zeros of the Chebyshev polynomial $T_{N+1}(\tau) - T_{N-1}(t)$. We analyze the stability and the convergence for the quadrature rule with a differentiable function. Also we show that the quadrature rule has an exponential convergence when the density function is analytic. © 2001 Academic Press

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## 1. INTRODUCTION

We consider a numerical evaluation of a Chebyshev approximation rule for the one sided finite part integral of the form

$$Q(f) = \fint_{-1}^{1} \frac{f(\tau)}{1-\tau} w(\tau) \, d\tau, \tag{1.1}$$

which has been extensively used in the application of boundary element techniques to the solution of three-dimensional elasticity problems and in solving the cylindrical wave equation (see [3] and [7]), where

$$w(\tau) = (1-\tau)^{\alpha} (1+\tau)^{\beta}, \qquad \beta > -1, \qquad -1 < \alpha < 0$$

and $f(t)$ is assumed to be smooth on $[-1, 1]$.

There exist many bibliographies on quadratures for the finite part integrals of (1.1). We mention the papers [13, 16, 17] based on polynomial interpolations. Recently, in [3, 4], Elliott considered the finite part integral

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(1.1) when $\beta = 0$, $-n - 1 < \alpha < -n$, and $n$ is nonnegative integer. In this case, the finite part integral (1.1) can be interpreted as the fractional derivative of order $\alpha$ of the funtion $f$ defined in $[-1, 1]$, at the point 1 (see [3, 15]). In [3], Elliott showed that the integration method based on the Bernstein interpolation polynomial with uniform abscissae converges to the exact result for all continuously differentiable functions. Also the Grünwald's algorithms [6] gives a simple algorithm evaluating the finite part integrals. However, the convergence rate is very slow for computational purposes. Recently, Elliott [4] made use of the Chebyshev polynomial to interpolate $f(\tau)$ instead of the Bernstein polynomial at the zeroes of the first kind Chebyshev polynomial $T_N(\tau)$. It is known [4] that the Chebyshev interpolation method [4] for (1.1) can yield, in general, better convergence than the Bernstein interpolation method [3], but the rule based on the Bernstein interpolation has better stability properties than that based on the zeros of $T_N(\tau)$.

In [10, 11, 12], we analyzed a trigonometric quadrature rule for a Cauchy type integrals based on the trigonometric interpolation polynomial at the practical abscissae. These method were originated from Clenshaw–Curtis method [1] for the integral $\int_{-1}^{1} f(\tau) \, d\tau$.

In this paper, we extend the Clenshaw–Curtis method for approximating the finite part integral (1.1). To that end, we first rewrite equation (1.1) as

$$Q(f) = I(f) + f(1) \, q_0,$$

where

$$I(f) = \int_{-1}^{1} \frac{f(\tau) - f(1)}{1 - \tau} \, w(\tau) \, d\tau. \tag{1.2}$$

and

$$q_0 = \begin{cases} \log 2 & \text{if } \alpha = \beta = 0, \\ 2^{\alpha + \beta} \dfrac{(\alpha + \beta + 1) \, \Gamma(\beta + 1) \, \Gamma(\alpha + 1)}{\alpha \Gamma(\alpha + \beta + 2)} & \text{if } \beta > -1, \quad -1 < \alpha < 0. \end{cases} \tag{1.3}$$

Then the integral $I(f)$ of (1.2) is no longer defined in the finite part sense.

In the Clenshaw–Curtis method [1], the function $f(\tau)$ is approximated by a sum of Chebyshev polynomials $T_k(\tau)$,

$$p_N^f(\tau) = \sum_{k=0}^{N}{}'' \, b_k^N \, T_k(\tau), \qquad -1 \leqslant \tau \leqslant 1, \tag{1.4}$$

where the coefficients $b_k^N$ are determined to satisfy the interpolation conditions

$$f(\tau_k^N) = p_N^f(\tau_k^N), \qquad \tau_k^N = \cos \frac{\pi k}{N}, \qquad 0 \leqslant k \leqslant N, \tag{1.5}$$

and given as follows [1]:

$$b_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' f(\tau_j^N) T_k(\tau_j^N), \qquad 0 \leqslant k \leqslant N. \tag{1.6}$$

A summation symbol with double primes denotes a sum with first and last terms halved.

We now approximate $f(\tau)$ and $f(1)$ of (1.2) by $p_N^f(\tau)$ and $p_N^f(1)$, respectively, and then introduce a new approximation $I_N(f)$ to the integral $I(f)$ as follows,

$$I_N(f) = I(p_N^f) = \sum_{k=0}^{N}{}'' b_k^N J_k, \tag{1.7}$$

where

$$J_k = \int_{-1}^{1} \frac{T_j(\tau) - T_j(1)}{1 - \tau} w(\tau) \, d\tau. \tag{1.8}$$

Alternatively, on using (1.4) and (1.6), the approximate integral $I_N(f)$ may be expressed as

$$I_N(f) = \sum_{k=0}^{N}{}'' f(\tau_k^N) \, \omega_k^N, \tag{1.9}$$

where

$$\omega_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' T_j(\tau_k^N) \, J_j. \tag{1.10}$$

Then, using the rule (1.7) or (1.9), we derive a new quadrature method for $Q(f)$ of the form

$$Q(f) = Q_N(f) + E_N(f), \tag{1.11}$$

where

$$Q_N(f) = I_N(f) + f(1) \, q_0, \tag{1.12}$$

where $q_0$ is given in (1.3). We shall call the rules $Q_N(f)$ and $I_N(f)$ as the Chebyshev quadrature rules.

Using the three term recurrence relation for $T_j(\tau)$, we show that $J_j$ of (1.8) can be explicitly calculated by a three term recurrence relation (see Section 2, (2.1)). Also, by using the trigonometric polynomial of the form: for $\tau$ not multiple of $2\pi$,

$$\sum_{j=0}^{n}{}' \cos j\tau = \frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{\tau}{2}}, \qquad n \geqslant 0, \tag{1.13}$$

which will be used several times in this paper, we show that the weights $\omega_k^N$ have the following expressions (see Lemma 2.1):

$$\omega_k^N = \begin{cases} \dfrac{(-1)^k}{N} \displaystyle\sum_{n=0}^{N}{}'' \, d_n \, \dfrac{T_{N-n}(1) - T_{N-n}(\tau_k^N)}{1-\tau_k^N}, & 0 < k \leqslant N, \\[4mm] -\dfrac{2}{N} \displaystyle\sum_{j=0}^{N-1}{}' \sum_{n=0}^{j}{}' \, (j-n+1)\, d_n, & k=0, \end{cases} \tag{1.14}$$

where

$$d_n = -2 \int_{-1}^{1} T_n(\tau) \, w(\tau) \, d\tau, \qquad n \geqslant 0,$$

which can be explicitly calculated (see Section 2, (2.2)). Define $\Lambda_N$ by

$$\Lambda_N = \sum_{k=0}^{N}{}'' \, |\omega_k^N|. \tag{1.15}$$

Then $\Lambda_N$ is very important numerically. Indeed, if $g$ is any function for which $Q(f)$ and $Q_N(f)$ exist, then from (1.12), we have

$$\begin{aligned} E_N(f) &= Q(f) - Q_N(f) \\ &= I(f) - I_N(f) \\ &= I(f-g) + I(g) - I_N(g) + I_N(g-f). \end{aligned}$$

If for a given $N$, we choose $g$ to be any polynomial $p_N$ of degree $\leqslant N$, then since $p_N^g(\tau) = g(\tau)$ and $I(g) - I_N(g) = I(g - p_N^g) = 0$, we have

$$E_N(f) = I(r_N) - I_N(r_N), \tag{1.16}$$

where $r_N(\tau) = f(\tau) - p_N(\tau)$. Now the Eq. (1.9) shows that

$$|I_N(r_N)| \leqslant M_N \Lambda_N, \tag{1.17}$$

where

$$M_N = \max_{\tau \in [-1, 1]} |r_N(\tau)| \qquad \text{and} \qquad \Lambda_N = \sum_{k=0}^{N}{}'' |\omega_k^N|. \tag{1.18}$$

The relations (1.16)–(1.18) show that the behaviour of the remainder $E_N(f)$ depends on those of $r_N(\tau)$ and $\Lambda_N$. Hence we shall call $\Lambda_N$ as the stability factor of the quadrature rule (1.12).

In Section 3, we show that the stability factor $\Lambda_N$ has the following behaviour:

$$\Lambda_N = \begin{cases} O(N^{-2\alpha}), & -1 < \alpha < -\dfrac{1}{2}, \\[2mm] O\left(N \log \dfrac{4N}{\pi}\right), & \alpha = -\dfrac{1}{2}, \\[2mm] O(N), & -\dfrac{1}{2} < \alpha < 0, \end{cases} \tag{1.19}$$

when $\beta = 0$. This shows that if $-1 < \alpha < -1/2$, then the present method has a better stability properties than the Elliott's method [4] based on the Chebyshev interpolation polynomial at the zeros of $T_N(\tau)$. Elliott has been obtained the following behaviour for the stability factor:

$$\Lambda_N = O(N^{-2\alpha} \log N).$$

The result (1.19), the relations (1.16)–(1.18) and some preliminary results for $r_N(\tau)$ of (1.16), are used to prove that the error $E_N(f)$ of (1.11) has the following behaviour (see Theorem 4.2):

$$|E_N(f)| = \begin{cases} O(N^{-2\alpha - p - \rho}), & -1 < \alpha < -\dfrac{1}{2}, \\[2mm] O\left(N^{1-p-\rho} \log \dfrac{4N}{\pi}\right), & \alpha = -\dfrac{1}{2} \\[2mm] O(N^{1-p-\rho}), & -\dfrac{1}{2} < \alpha < 0 \end{cases} \tag{1.20}$$

when the function $f(\tau)$ has continuous derivative up to order $p \geqslant 1$ and its derivative $f^{(p)}(\tau)$ satisfies Hölder continuity of order $\rho$, $0 < \rho \leqslant 1$. This result shows that the present method are substantially improved compared with the one of Elliott's method (see [4, Theorem 4.6]).

Finally, in Section 5, we will show that the error $E_N(f)$ of (1.11) has an exponential convergence rate when the function $f$ is an analytic function in $[-1, 1]$ (see Theorem 5.2).

## 2. PRACTICAL IMPLEMENTATION

The first task in practice is to calculate the sequence $\{J_j\}$ defined by (1.8). To that end, we make use of the three-term recurrence relation for the Chebyshev polynomial and the definition of Beta function. From the recurrence relation of the Chebyshev polynomial, we see that

$$\frac{T_{j+2}(\tau) - T_{j+2}(1)}{\tau - 1} = 2\frac{T_{j+1}(\tau) - T_{j+1}(1)}{\tau - 1} - \frac{T_j(\tau) - T_j(1)}{\tau - 1} + 2T_{j+1}(\tau)$$

and hence

$$J_{j+2} - 2J_{j+1} + J_j = d_{j+1}, \qquad k \geqslant 2, \tag{2.1}$$

where $d_j$ are defined in (1.14). Applying the change of variable $\tau = \cos y$ to the integrals $d_j$ and using the definition of the Beta function, we can have the following expressions for $d_j$ [10],

$$d_j = -2^{\alpha + \beta} \sum_{i=0}^{j} \binom{2j}{2i} (-1)^{j-i} \frac{\Gamma(j - i + \alpha + 1)\,\Gamma(i + \beta + 1)}{\Gamma(j + \alpha + \beta + 2)}, \qquad j \geqslant 0, \tag{2.2}$$

where $\Gamma(\cdot)$ denote the Gamma function. We easily see that

$$J_0 = 0 \tag{2.3}$$

and the definition of $\{d_j\}$ shows that

$$J_1 = -\int_{-1}^{1} w(\tau)\, d\tau = \frac{d_0}{2}. \tag{2.4}$$

Hence, once $\{d_j\}$ is computed by the formula (2.2), then we can calculate $\{J_j\}$ by using the recurrence relation (2.1) with the initial conditions (2.3) and (2.4).

The next step, given the function $f$ is to choose $N$ and then to calculate the function values $f(\tau_j^N)$, $0 \leqslant j \leqslant N$. Note that there is a significant economy if the value of $N$ is doubled progressively; all previous function values can then be reused, because

$$f(\tau_j^N) = f(\tau_{2j}^{2N}) .$$

We assume that $Q_N f$ is to be calculated by the formula (1.12). Then the next step is to calculated the coefficients $b_j^N$ from (1.6). The Chebyshev series may be calculated by using the Clenshaw recurrence method [1]. An alternative procedure for evaluating the coefficients $b_j^N$ is the Fast Fourier Transformation method (FFT) of [5]. That latter procedure will certainly be the most efficient if $N$ is very large.

When the coefficients $b_j^N$ are known, the approximate integral $Q_N f$ is calculated using (1.7) and (1.12). If the error so obtained is too large to be acceptable, then the calculation can be repeated by doubling $N$. It is advantageous to have more chances of checking the stopping criterion than by doubling $N$, in order to enhance the efficiency of automatic quadrature. In [19], Hasegawa *et al.* allowed $N$ to take the forms $3 \times 2^n$ and $5 \times 2^n$ as well as $2^n$, that is,

$$N = 3, 4, 5, ..., 3 \times 2^n, 4 \times 2^n, 5 \times 2^n, ... \qquad (n = 1, 2, ...). \qquad (2.5)$$

For a detail construction of $\{p_N^f(\tau)\}$ by increasing $N$ as in (2.5), we refer to the paper [19].

In the rest of this section we consider the weights $\omega_k^N$ of (1.10).

LEMMA 2.1. *Let* $\{\omega_k^N\}$ *be the weights defined in* (1.10) *and* $d_n$ *the integrals defined in* (1.14). *Then we have that*

$$\omega_k^N = \frac{(-1)^k}{N} \sum_{n=0}^{N}{}'' d_n \frac{T_{N-n}(1) - T_{N-n}(\tau_k^N)}{1 - \tau_k^N} , \qquad 0 < k \leqslant N \qquad (2.6)$$

*and*

$$\omega_0^N = -\frac{2}{N} \sum_{j=0}^{N-1}{}' \sum_{n=0}^{j}{}' (j - n + 1) d_n, \qquad (2.7)$$

*where* $\tau_k^N$ *are the practical abscissae defined in* (1.5).

*Proof.*  For showing (2.6), we let $x_k = \cos^{-1} \tau_k^N$ and $x = \cos^{-1} \tau$. Then, using the formula $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$, we find that

$$l_k^N(\tau) := \frac{2}{N} \sum_{j=0}^{N}{}'' T_j(\tau_k^N) \, T_j(\tau)$$

$$= \frac{1}{N} \sum_{j=0}^{N}{}'' (\cos j(x + x_k) + \cos j(x - x_k)).$$

Now from the formula (1.13), we can show that

$$l_k^N(\tau) = \frac{\left( \begin{array}{c} \cos \dfrac{x + x_k}{2} \sin \dfrac{x - x_k}{2} \sin N(x + x_k) \\[2mm] + \sin \dfrac{x + x_k}{2} \cos \dfrac{x - x_k}{2} \sin N(x - x_k) \end{array} \right)}{N(\cos x_k - \cos x)}.$$

Since $x_k = \pi k/N$,

$$\sin N(x \pm x_k) = (-1)^k \sin Nx$$

whence we have that

$$l_k^N(\tau) = \frac{(-1)^k}{N} \frac{\sin Nx \sin x}{\cos x_k - \cos x}$$

$$= \frac{(-1)^k}{N} \frac{U_{N-1}(\tau)(1 - \tau^2)}{\tau_k^N - \tau}$$

$$= \frac{(-1)^k}{2N} \frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{\tau_k^N - \tau}, \tag{2.8}$$

where $U_n(\tau)$ are the second kind of the Chabyshev polynomial. This Eq. (2.8) shows that the weights $\omega_k^N$ of (1.10) can be written as

$$\omega_k^N = \frac{(-1)^k}{2N} \int_{-1}^{1} \frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{(\tau_k^N - \tau)(1 - \tau)} w(\tau) \, d\tau$$

$$= \frac{(-1)^k}{2N} \frac{1}{\tau_k^N - 1} \int_{-1}^{1} w(\tau) \left( \frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{1 - \tau} \right.$$

$$\left. - \frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{\tau_k^N - \tau} \right) d\tau. \tag{2.9}$$

Since $\tau_k^N$ are the zero points of the polynomial of $T_{N-1}(\tau) - T_{N+1}(\tau)$, we can have that for any $0 \leqslant k \leqslant N$,

$$\frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{\tau_k^N - \tau}$$

$$= \frac{T_{N-1}(\tau) - T_{N-1}(\tau_k^N)}{\tau_k^N - \tau} + \frac{T_{N+1}(\tau_k^N) - T_{N+1}(\tau)}{\tau_k^N - \tau}. \qquad (2.10)$$

Elliott [2] gives the following identity involving the second kind of the Chebyshev polynomial:

$$\frac{T_{k+1}(\tau) - T_{k+1}(\tau_k^N)}{\tau - \tau_k^N} = 2 \sum_{n=0}^{k}{}' U_{k-n}(\tau_k^N) \, T_n(\tau), \qquad k \geqslant 0. \qquad (2.11)$$

Using the identities $U_k(\tau) - U_{k-2}(\tau) = 2T_k(\tau)$, $(k \geqslant 1)$, where we define $U_{-1}(\tau) = 0$, and (2.10) and (2.11) give

$$\frac{T_{N-1}(\tau) - T_{N+1}(\tau)}{\tau_k^N - \tau} = 4 \sum_{n=0}^{N}{}'' T_{N-n}(\tau_k^N) \, T_n(\tau). \qquad (2.12)$$

Combining (2.9) and (2.12) and using the definition of $d_n$ of (1.14), we have the identity (2.6). To show (2.7), we note that from (2.11),

$$\frac{T_{j+1}(\tau) - 1}{\tau - 1} = 2 \sum_{n=0}^{j}{}' U_{j-n}(1) \, T_n(\tau), \qquad j \geqslant 0. \qquad (2.13)$$

Now using (1.10), (2.8) and (2.13), we can write $\omega_0^N$ as follows:

$$\omega_0^N = \frac{2}{N} \sum_{j=0}^{N}{}'' \int_{-1}^{1} \frac{T_j(\tau) - T_j(1)}{1 - \tau} w(\tau) \, d\tau$$

$$= \frac{4}{N} \sum_{j=0}^{N-1}{}' \sum_{n=0}^{j}{}' U_{j-n}(1) \int_{-1}^{1} T_n(\tau) \, w(\tau) \, d\tau$$

Hence using the definition of $d_n$ of (1.14), we can complete the proof.  ∎

## 3. ASYMPTOTIC BEHAVIOUR FOR $\Lambda_N$

In this section, we shall estimate the asymptotic behaviour for the stability factor $\Lambda_N$ of (1.18) using the expression (2.6) and (2.7) for the

weights $\omega_k^N$. For simplicity of our analysis, we assume that $\beta = 0$ in the weight function of (1.1), throughout this section.

THEOREM 3.1. *Let $\omega_k^N$ be weights defined in* (1.10) *and let $\{d_j\}$ be a sequence defined in* (1.14). *Assume that $-1 < \alpha < 0$ and $\beta = 0$. Then we have that*

$$|\omega_0^N| \leqslant c_1(\alpha) \, N^{-2\alpha}, \tag{3.1}$$

*where*

$$c_1(\alpha) = \frac{2^\alpha \pi}{1 + \alpha} + \frac{\pi^{2\alpha} 2^{2-\alpha}}{|\alpha|}.$$

*Proof.* From the definition of $d_j$ of (1.14) and the expression (2.7), we find that

$$\omega_0^N = \frac{4}{N} \, \sideset{}{'}\sum_{j=0}^{N-1} \, \sideset{}{'}\sum_{n=0}^{j} (j+1-n) \int_{-1}^{1} T_n(\tau) \, w(\tau) \, d\tau$$

(using the change of variable $\tau = \cos y$)

$$= \frac{2^{\alpha+3}}{N} \int_0^\pi \sin^{2\alpha+1} \frac{y}{2} \cos \frac{y}{2} \, g(y) \, dy, \tag{3.2}$$

where

$$g(y) = \sideset{}{'}\sum_{j=0}^{N-1} \left( (j+1) \sideset{}{'}\sum_{n=0}^{j} \cos ny - \sum_{n=1}^{j} n \cos ny \right).$$

Using the following identities,

$$\sideset{}{'}\sum_{n=0}^{j} \cos ny = \frac{\sin \left( j + \frac{1}{2} \right) y}{2 \sin \frac{y}{2}},$$

$$\sum_{n=1}^{j} n \cos ny = \frac{(j+1) \sin \left( j + \frac{1}{2} \right) y}{2 \sin \frac{y}{2}} - \frac{1 - \cos(j+1) y}{4 \sin^2 \frac{y}{2}}, \tag{3.3}$$

the function $g(y)$ given in (3.2) can be written as

$$g(y) = \frac{\csc^2 \frac{y}{2}}{4} \sum_{j=0}^{N-1}{}' (1 - \cos(j+1) y)$$

$$= \frac{\csc^3 \frac{y}{2}}{8} \left( 2N \sin \frac{y}{2} - \sin Ny \cos \frac{y}{2} \right). \tag{3.4}$$

Substituting (3.4) into (3.2) gives

$$\omega_0^N = \frac{2^\alpha}{N} \int_0^\pi \sin^{2\alpha-2} \frac{y}{2} \cos \frac{y}{2} \left( 2N \sin \frac{y}{2} - \sin Ny \cos \frac{y}{2} \right) dy$$

(using the change of variable $y = 2x$)

$$= \frac{2^{\alpha+1}}{N} \int_0^{\pi/2} \sin^{2\alpha-2} y \cos y (2N \sin y - \sin 2Ny \cos y) \, dy$$

$$= A + B, \tag{3.5}$$

where

$$A = \frac{2^{\alpha+1}}{N} \int_0^{\pi/(2N)} \sin^{2\alpha-2} y \cos y (2N \sin y - \sin 2Ny \cos y) \, dy$$

$$B = \frac{2^{\alpha+1}}{N} \int_{\pi/(2N)}^{\pi/2} \sin^{2\alpha-2} y \cos y (2N \sin y - \sin 2Ny \cos y) \, dy.$$

For estimating $A$, we note that

$$\frac{2N \sin y - \sin 2Ny \cos y}{\sin^3 y} \leqslant \frac{2}{3} (N + 2N^3), \qquad y \in \left[ 0, \frac{\pi}{2N} \right].$$

Hence we have that

$$A \leqslant \frac{2^{\alpha+2}(N + 2N^3)}{3N} \int_0^{\pi/(2N)} \sin^{2\alpha+1} y \, dy$$

$$\left( \text{using the fact } \frac{2y}{\pi} \leqslant \sin y \leqslant y \right.$$

$$\left. \text{and } \left( \frac{2y}{\pi} \right)^{2\alpha+1} \geqslant 1, \text{ when } -1 < \alpha < -\frac{1}{2} \right)$$

$$\leqslant \frac{2^{\alpha+2}(N+2N^3)}{3N} \int_0^{\pi/(2N)} \left(\frac{2y}{\pi}\right)^{2\alpha+1} dy$$

$$= \frac{2^\alpha \pi}{3(1+\alpha)} N^{-2\alpha} \left(\frac{1}{N^2} + 2\right)$$

$$\leqslant \frac{2^\alpha \pi}{1+\alpha} N^{-2\alpha}. \tag{3.6}$$

Since

$$\left| \frac{2N \sin y - \sin 2Ny \cos y}{\sin y} \right| \leqslant 4N, \qquad y \in \left[0, \frac{\pi}{2}\right],$$

$B$ can be bounded as follows:

$$B \leqslant 2^{\alpha+3} \int_{\pi/(2N)}^{\pi/2} \sin^{2\alpha-1} y \cos y \, dy$$

$$= \frac{2^{\alpha+2}}{\alpha} \left(1 - \left(\frac{\pi}{2N}\right)^{2\alpha}\right)$$

$$\leqslant N^{-2\alpha} \frac{\pi^{2\alpha} 2^{2-\alpha}}{|\alpha|}. \tag{3.7}$$

Finally, substituting (3.6) and (3.7)) into (3.5) gives the desired result (3.1).

Using this Theorem 3.1, the expression (2.6) for $\omega_k^N$ and the definition of $\{d_j\}$ defined in (1.14), we can obtain the following behaviour for the stability factor $\Lambda_N$.

THEOREM 3.2. *Let $\omega_k^N$ be weights defined in* (1.10) *and let $\{d_j\}$ be a sequence defined in* (1.14). *Assume that $-1 < \alpha < 0$ and $\beta = 0$. Then the stability factor $\Lambda_N$ of the rule* (1.11) *satisfies*

$$\Lambda_N \leqslant C \begin{cases} CN^{-2\alpha}, & -1 < \alpha < -\dfrac{1}{2}, \\ N \log \dfrac{4N}{\pi}, & \alpha = -\dfrac{1}{2}, \\ N, & -\dfrac{1}{2} < \alpha < 0, \end{cases} \tag{3.8}$$

*where $C$ is a constant depending only on $\alpha$.*

*Proof.* The formulae (2.6) and the definition of $d_n$ given in (1.14) show that

$$\omega_k^N = \frac{(-1)^k}{N(1-\tau_k^N)}\left(\sum_{n=0}^{N}{}'' d_n - (-1)^k \sum_{n=0}^{N}{}'' d_n \cos\frac{\pi k n}{N}\right)$$

(using the change of variable $\tau = \cos y$)

$$= -\frac{2^{\alpha+2}(-1)^k}{N(1-\tau_k^N)}\int_0^\pi \sin^{2\alpha+1}\frac{y}{2}\cos\frac{y}{2}\sum_{n=0}^{N}{}''\cos ny\left(1-(-1)^k\cos\frac{\pi k n}{N}\right)dy$$

$$= -\frac{2^{\alpha+2}(-1)^k}{N(1-\tau_k^N)}\left(\int_0^\pi \sin^{2\alpha+1}\frac{y}{2}\cos\frac{y}{2}\frac{\sin Ny\cos\frac{y}{2}}{2\sin\frac{y}{2}}dy\right.$$

$$\left.-\frac{1}{2}\int_0^\pi \sin^{2\alpha+1}\frac{y}{2}\cos\frac{y}{2}\frac{\sin Ny\sin y}{\tau_k^N-\cos y}dy\right)$$

$$= \frac{2^{\alpha+1}}{N}\int_0^\pi \sin^{2\alpha}\frac{y}{2}\cos\frac{y}{2}\sin Ny\frac{1}{\tau_k^N-\cos y}dy.$$

Hence we have

$$\sum_{k=1}^{N}{}' |\omega_k^N| \leqslant A+B, \tag{3.9}$$

where

$$A = \frac{2^{\alpha+1}}{N}\int_0^{\pi/(2N)}\sin^{2\alpha}\frac{y}{2}\cos\frac{y}{2}\sin Ny\sum_{k=1}^{N}\frac{1}{\cos y-\tau_k^N}dy,$$

$$B = \frac{2^{\alpha+1}}{N}\int_{\pi/(2N)}^{\pi}\sin^{2\alpha}\frac{y}{2}\cos\frac{y}{2}|\sin Ny|\sum_{k=1}^{N}\frac{1}{|\tau_k^N-\cos y|}dy.$$

If we let $t = \cos y$ and $t_k = \tau_k^N$, then since $t_k$ are the zeros of $T_{N+1}(t) - T_{N-1}(t)$, we have

$$\sum_{k=0}^{N}\frac{1}{t-t_k} = \frac{(N+1)U_N(t)-(N-1)U_{N-2}(z)}{T_{N+1}(t)-T_{N-1}(t)}$$

whence

$$\sum_{k=1}^{N} \frac{1}{\cos y - \tau_k^N} = \frac{\csc^2 \dfrac{y}{2}}{2} - (\cot y + N \cot Ny) \csc y$$

$$\leqslant \frac{2N^2 + 1}{6}, \qquad y \in \left[ 0, \frac{\pi}{2N} \right].$$

Thus, we find that

$$A \leqslant \frac{(2N^2 + 1)\, 2^{\alpha + 1}}{6N} \int_0^{\pi/(2N)} \sin^{2\alpha + 1} \frac{y}{2} \frac{\sin Ny}{\sin \dfrac{y}{2}} \, dy$$

$$\leqslant \frac{(2N^2 + 3)\, 2^{\alpha + 1}}{6} \int_0^{\pi/(4N)} \sin^{2\alpha + 1} y \, dy$$

(using the same technique with (3.6))

$$\leqslant \frac{(2N^2 + 3)\, 2^{\alpha + 1}}{6} \int_0^{\pi/(4N)} \left( \frac{2y}{\pi} \right)^{2\alpha + 1} dy$$

$$= \frac{2^{-4-a} N^{-2(1+\alpha)} (3 + 2N^2)\, \pi}{3(1+\alpha)}$$

$$\leqslant \frac{\pi}{(1+\alpha)\, 2^{4+\alpha}} N^{-2\alpha}, \qquad \text{if} \quad N \geqslant 2. \tag{3.10}$$

For estimating $B$ of (3.9), we make use of the mean value theorem for the integral and the change of variable $y/2 = x$. Then we have

$$B = \frac{2^{\alpha + 1}}{N} \sum_{k=1}^{N} \left| \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\tau_k^N - \cos \theta} \right| \int_{\pi/(4N)}^{\pi/2} \sin^{2\alpha} x \cos x \, dx,$$

$$\text{for some } \theta \in \left[ \frac{\pi}{2N}, \pi \right]$$

$$= B_1 \begin{cases} \dfrac{1}{1+2\alpha} \left( 1 - \sin^{1+2\alpha} \dfrac{\pi}{4N} \right), & \alpha \neq -\dfrac{1}{2} \\[2mm] -\log \sin \dfrac{\pi}{4N}, & \alpha = -\dfrac{1}{2} \end{cases}$$

$$\leqslant B_1 \begin{cases} \dfrac{1}{|2\alpha+1| \, 2^{2\alpha+1}} \, N^{-2\alpha-1}, & -1 < \alpha < -\dfrac{1}{2}, \\[2mm] \log \dfrac{4N}{\pi}, & \alpha = -\dfrac{1}{2}, \\[2mm] \dfrac{1}{2\alpha+1}, & -\dfrac{1}{2} < \alpha < 0, \end{cases}$$

(3.11)

where

$$B_1 = \frac{1}{N} \sum_{k=1}^{N} \left| \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\tau_k^N - \cos \theta} \right|.$$

For estimating $B_1$, we assume that $\theta \in (\frac{\pi}{2N}, \frac{\pi}{N})$ or $\theta \in (\frac{l\pi}{N}, \frac{(l+1)\pi}{N})$ for some fixed $l$, $(1 \leqslant l < \frac{N}{2}.)$ We first consider the case $\theta \in (\frac{\pi}{2N}, \frac{\pi}{N})$. In this case, we can estimate $B_1$ as follows:

$$B_1 \leqslant \frac{1}{N} \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\cos \theta - \cos \dfrac{\pi}{N}} + \frac{1}{\pi} \int_{\pi/N}^{\pi} \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\cos \theta - \cos y} \, dy$$

$$= \frac{1}{N} \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\cos \theta - \cos \dfrac{\pi}{N}} + \frac{1}{\pi} \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\sin \theta} \log \left| \frac{\sin(\theta + \pi/N)/2}{\sin(\theta - \pi/N)/2} \right|$$

$$= O(N). \tag{3.12}$$

We now assume that $\theta \in (\frac{l\pi}{N}, \frac{(l+1)\pi}{N})$ for some fixed $l$, $(1 \leqslant l < \frac{N}{2})$ and split the sum $B_1$ as three parts,

$$B_1 = B_1^1 + B_1^2 + B_1^3, \tag{3.13}$$

where

$$B_1^1 = \frac{\left| \sin N\theta \cos \dfrac{\theta}{2} \right|}{N} \left( \frac{1}{|\cos \pi l/N - \cos \theta|} + \frac{1}{|\cos \pi(l+1)/N - \cos \theta|} \right),$$

$$B_1^2 = \frac{1}{N} \sum_{k=1}^{l-1} \left| \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\tau_k^N - \cos \theta} \right|,$$

$$B_1^3 = \frac{1}{N} \sum_{k=l+2}^{N} \left| \frac{\sin N\theta \cos \dfrac{\theta}{2}}{\tau_k^N - \cos \theta} \right|.$$

Since $\theta \in (\frac{l\pi}{N}, \frac{(l+1)\pi}{N})$, by using the Cauchy's theorem and the fact, we find that

$$B_1^1 \leqslant \left| \frac{\cos N\chi}{\sin \chi} \right| + \left| \frac{\cos N\chi'}{\sin \chi'} \right|, \qquad \text{for some} \quad \chi, \chi' \in \left( \frac{\pi l}{N}, \frac{\pi(l+1)}{N} \right)$$

$$\leqslant \left| \frac{1}{\sin \chi} \right| + \left| \frac{1}{\sin \chi'} \right|$$

$$= O(N). \tag{3.14}$$

The summation $B_1^2$ and $B_1^3$ can be estimated by making use of the lower sum of the Riemann integral. We first consider the summation $B_1^2$. Since $\cos y - \cos \theta$ is a decreasing function on $[\pi/N, \pi l/N]$, we see that

$$B_1^2 \leqslant \frac{1}{\pi} \int_{\pi/N}^{\pi l/N} \frac{\left| \sin N\theta \cos \dfrac{\theta}{2} \right|}{\cos y - \cos \theta} \, dy$$

$$= \frac{\left| \sin N\theta \cos \dfrac{\theta}{2} \right|}{\pi \sin \theta} \log \left| \frac{\sin[(\theta + \pi l/N)/2] \sin[(\theta - \pi/N)/2]}{\sin[(\theta - \pi l/N)/2] \sin[(\theta + \pi/N)/2]} \right|$$

$$= O(N). \tag{3.15}$$

Since $\cos \theta - \cos y$ is an increasing function on $[\pi(l+1)/N, \pi]$, we have that

$$B_1^3 \leqslant \frac{1}{\pi} \int_{\pi(l+1)/N}^{\pi} \frac{\left| \sin N\theta \cos \dfrac{\theta}{2} \right|}{\cos \theta - \cos y} \, dy$$

$$= \frac{\left| \sin N\theta \cos \dfrac{\theta}{2} \right|}{\pi \sin \theta} \log \left| \frac{\sin[(\theta + \pi(l+1)/N)/2]}{\sin[(\theta - \pi(l+1)/N)/2]} \right|$$

$$= O(N). \tag{3.16}$$

Summarizing (3.11)–(3.16), we have that

$$B \leqslant C \begin{cases} N^{-2\alpha}, & -1 < \alpha < -\dfrac{1}{2}, \\[2mm] N \log \dfrac{4N}{\pi}, & \alpha = -\dfrac{1}{2}, \\[2mm] N, & -\dfrac{1}{2} < \alpha < 0. \end{cases} \tag{3.17}$$

Finally, combining (3.10) and (3.16), then from (3.9) and Theorem 3.1, we can complete the proof. ∎

*Remark* 3.1. In [4], Elliott showed that the stability factor $\Lambda_N$ for the rule based on the classical abscissae has the behaviour of the form:

$$\Lambda_N = O(N^{-2\alpha} \log N).$$

Hence from (1.17) and (3.8), we see that the Chebyshev interpolation method based on the practical abscissae has a better stability properties than that based on the classical abscissae.

## 4. CONVERGENCE RESULTS FOR CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

In this section we shall derive a bound for the error $|E_N(f)|$ when the function $f$ is differentiable and its derivative is Hölder continuous. To that

end, we quote the following known results. For the proof, we refer to the reference [9, 14, 18].

LEMMA 4.1. *Suppose the function $f(\tau)$ possesses continuous derivatives up to order $p \geqslant 1$ and the derivative $f^{(p)}(\tau)$ satisfies Hölder continuity of order $\rho$. Then there exists a polynomial $p_N(\tau)$ of order $N$ such that*

$$\max_{\tau \in [-1, 1]} |f(\tau) - p_N(\tau)| \leqslant \frac{M_0}{N^{p+\rho}} \quad and \quad |p_N(\tau) - p_N(t)| \leqslant N_0 |\tau - t|, \tag{4.1}$$

*where $M_0$ and $N_0$ are constants independent of $N$ and $\tau$ and $t$. Further, for $r_N(\tau) = f(\tau) - p_N(\tau)$, we have*

$$|r_N^{(k)}(\tau)| = O(N^{-p-\rho+k}), \qquad k = 1, 2, .... \tag{4.2}$$

Using this Lemma 4.1 and the previous Theorem 3.2, we have the following error bound for the quadrature rule (1.12).

THEOREM 4.1. *Let us consider the quadrature rule (1.11). Suppose the function $f(\tau)$ possesses continuous derivatives up to order $p \geqslant 1$ and the derivative $f^{(p)}(\tau)$ satisfies Hölder continuity of order $\rho$. Then the remainder term $E_N(f) = Q(f) - Q_N(f)$ satisfies*

$$|E_N(f)| = \begin{cases} O(N^{-2\alpha-p-\rho}), & -1 < \alpha < -\frac{1}{2}, \\ O\left(N^{1-p-\rho} \log \frac{4N}{\pi}\right), & \alpha = -\frac{1}{2}, \\ O(N^{1-p-\rho}), & -\frac{1}{2} < \alpha < 0. \end{cases} \tag{4.3}$$

*Proof.* By the facts (1.16), Theorem 3.2 and Lemma 4.1, we can easily prove the theorem. Indeed, for $r_N(\tau) = f(\tau) - p_N(\tau)$, the fact (1.16) shows that

$$|E_N(f)| \leqslant |I(r_N)| + \max_{\tau \in [-1, 1]} |r_N(\tau)| \sum_{k=0}^{N}{}'' |w_k^N|$$

$$\leqslant A_1 + A_2, \tag{4.4}$$

where

$$A_1 = \int_{-1}^{1} w(\tau) \left| \frac{r_N(\tau) - r_N(1)}{1 - \tau} \right| d\tau, \qquad A_2 = \max_{\tau \in [-1, 1]} |r_N(\tau)| \sum_{k=0}^{N} {}'' |w_k^N|.$$

The inequality (4.1) and the result of Theorem 3.2 show that the term $A_2$ has the behaviour

$$A_2 = \begin{cases} O(N^{-p-\rho-2\alpha}), & -1 < \alpha < -\frac{1}{2}, \\[2mm] O\left(N^{-p-\rho+1} \log \frac{4N}{\pi}\right), & \alpha = -\frac{1}{2}, \\[2mm] O(N^{-p-\rho+1}), & -\frac{1}{2} < \alpha < 0. \end{cases} \tag{4.5}$$

For estimating $A_1$, we use the mean value theorem and the fact (4.2). Then we have

$$\begin{aligned} A_1 &= \int_{-1}^{1} w(\tau) \, |r'_N(\chi)| \, d\tau \\[2mm] &\leqslant \max_{\tau \in [-1, 1]} |r'_N(\tau)| \int_{-1}^{1} w(\tau) \, d\tau \\[2mm] &= O(N^{-p-\rho+1}) \int_{-1}^{1} w(\tau) \, d\tau. \end{aligned} \tag{4.6}$$

Therefore, substituting the asymptotic behaviours (4.5) and (4.6) into (4.4), we have the desired result (4.3). ∎

## 5. CONVERGENCE RESULTS FOR ANALYTIC FUNCTIONS

In this section we shall derive error bounds for the proposed quadrature rule of (1.12) when the function $f$ is analytic. Let $\varepsilon_\kappa$ denote the ellipse in the complex $z = x + iy$ with foci $(x, y) = (-1, 0)$, $(1, 0)$ and semimajor axis $a = \frac{1}{2}(\kappa + \kappa^{-1})$ and semiminor axis $b = \frac{1}{2}(\kappa - \kappa^{-1})$ for a constant $\kappa > 1$.

Assume that $f(z)$ is single-valued and analytic inside and on $\varepsilon_\kappa$. Then, by using the Cauchy's formula for the function $f$ and the series expansion

$$\sum_{k=0}^{\infty} {}' \frac{T_k(\tau)}{(z + \sqrt{z^2 - 1})^k} = \frac{\sqrt{z^2 - 1}}{2(z - \tau)}$$

we can have the following series expression for the remainder of the interpolation error [8, (3.2)],

$$f(\tau) - p_N^f(\tau) = \frac{1}{2\pi i} \oint_{\varepsilon_\kappa} \frac{\omega_{N+1}(\tau) f(z)}{(z-\tau)\,\omega_{N+1}(z)}\,dz$$

$$= \omega_{N+1}(\tau) \sum_{k=0}^{\infty}{}' V_k^N(f)\, T_k(\tau), \qquad (5.1)$$

where $\omega_{N+1}(\tau) = T_{N+1}(\tau) - T_{N-1}(\tau) = 2(\tau^2 - 1)\, U_{N-1}(\tau)$, $N \geqslant 1$ and

$$V_k^N(f) = \frac{1}{\pi i} \oint_{\varepsilon_\kappa} \frac{f(z)}{\omega_{N+1}(z)\,\sqrt{z^2-1}\,(z+\sqrt{z^2-1})^k}\,dz, \qquad k \geqslant 0. \tag{5.2}$$

Thus, from (1.11) and (5.1) we obtain an explicit expression for the error in the approximate integration rule (1.12) as follows,

$$E_N(f) = I(f - p_N^f) = \sum_{k=0}^{\infty}{}' V_k^N(f)\, \Omega_k^N, \tag{5.3}$$

where $\Omega_k^N = I(\omega_{N+1} T_k)$. Hence for estimating $|E_N(f)|$, we need the behaviours for $V_k^N(f)$ and $\Omega_k^N$. We first consider the terms $V_k^N(f)$ given in (5.2). To that end, we assume that $f(z)$ is a meromorphic function which has $M$ simple poles at the points $z_m$ ($m = 1, 2, ..., M$) outside of $\varepsilon_\kappa$ with residues $Resf(z_m)$. Then performing the contour integral (5.2) gives

$$|V_k^N(f)| \leqslant \kappa_*^{-k} |V_0^N(f)| = O(\kappa_*^{-k-N}) \tag{5.4}$$

and

$$|V_0^N(f)| \sim |b_N^N| \frac{\kappa_*}{\kappa_*^2 - 1}, \tag{5.5}$$

where $\kappa_* = \min_{1 \leqslant m \leqslant M} |z_m + \sqrt{z_m^2 - 1}| > 1$. For a detail proof, we refer the paper [8] and reference there in. We now consider the terms $\Omega_k^N$ of (5.3).

By using the relation

$$2T_m(t)\, T_n(t) = T_{n+m}(t) + T_{|n-m|}(t), \qquad n, m \geqslant 0 \tag{5.6}$$

and the definition of $\Omega_k^N$ in (5.3), it follows that

$$2\Omega_k^N = \int_{-1}^1 \left( \frac{T_{N+k+1}(\tau) - T_{N+k-1}(\tau)}{1-\tau} \right.$$

$$\left. \pm \frac{T_{|N-k|+1}(\tau) - T_{|N-k|-1}(\tau)}{1-\tau} \right) w(\tau)\, d\tau, \qquad k \geqslant 0, \qquad (5.7)$$

where the plus sign is taken if $N - k \geqslant 1$ and the minus sign if $k - N \geqslant 1$. Further, the second integrand of (5.7) should be ignored with $N = k$. Now applying the formula (2.10) to the Eq. (5.7), we have

$$\Omega_k^N = \begin{cases} 2 \displaystyle\sum_{n=0}^{N+k}{}'' \int_{-1}^1 T_n(\tau)\, w(\tau)\, d\tau \\ \qquad \pm 2 \displaystyle\sum_{n=0}^{|N-k|}{}'' \int_{-1}^1 T_n(\tau)\, w(\tau)\, d\tau, \qquad |N-k| \geqslant 1, \\ 2 \displaystyle\sum_{n=0}^{N+k}{}'' \int_{-1}^1 T_n(\tau)\, w(\tau)\, d\tau, \qquad\qquad N = k. \end{cases}$$

$$(5.8)$$

Thus we can see that the terms $\Omega_k^N$ are bounded independent of $k$ as follows:

LEMMA 5.1. *Let $\Omega_k^N$ be defined by (5.3) and let $-1 < \alpha < 0$ and $-1 < \beta$ be fixed. Then we have, for any $k \geqslant 1$,*

$$|\Omega_k^N| \leqslant \begin{cases} N, & -1 < \alpha \leqslant -\frac{1}{2}, \\ 1, & -\frac{1}{2} < \alpha < 0. \end{cases} \qquad (5.9)$$

*Proof.* Applying the change of variable $\tau = \cos y$ to the integrals in (5.8), we have

$$\Omega_k^N = 2^{\alpha+2} \begin{cases} \displaystyle\int_0^\pi \sin^{2\alpha+1}\frac{y}{2} \cos^{2\beta+1}\frac{y}{2} \\ \qquad \times \left( \displaystyle\sum_{n=0}^{N+k}{}'' \cos ny \pm \sum_{n=0}^{|N-k|}{}'' \cos by \right) dy, \qquad N \neq k, \\ \displaystyle\int_0^\pi \sin^{2\alpha+1}\frac{y}{2} \cos^{2\beta+1}\frac{y}{2} \sum_{n=0}^{N+k}{}'' \cos ny\, dy, \qquad N = k. \end{cases}$$

$$(5.10)$$

From the formula (1.13), we note that

$$\sum_{n=0}^{m}{}'' \cos ny = \frac{\sin 2my}{2 \sin \frac{y}{2}}.$$

Hence the expression (5.10) for $\Omega_k^N$ can be written as follows:

$$\Omega_k^N = 2^{\alpha+1} \begin{cases} 2 \int_0^\pi \sin^{2\alpha+1} \frac{y}{2} \cos^{2\beta+1} \frac{y}{2} \cos 2ky \, \frac{\sin 2Ny}{\sin \frac{y}{2}} \, dy, & N \neq k \\[2ex] \int_0^\pi \sin^{2\alpha+1} \frac{y}{2} \cos^{2\beta+1} \frac{y}{2} \frac{\sin 4Ny}{\sin \frac{y}{2}} \, dy, & N = k \end{cases}$$

whence we can easily get the bound (5.9). ∎

Combining (5.3) and (5.9), the bound of $|E_N(f)|$ becomes

$$|E_N f| \leqslant H(\alpha; N) \sum_{k=0}^{\infty}{}' |V_k^N(f)|, \tag{5.11}$$

where $H(\alpha; N)$ is defined by

$$H(\alpha; N) = \begin{cases} N, & -1 < \alpha \leqslant -\frac{1}{2}, \\ 1, & -\frac{1}{2} < \alpha < 0. \end{cases} \tag{5.12}$$

Finally the asymptotic behaviour of $|V_k^N(f)|$ given in (5.4) and (5.5) gives that

$$\begin{aligned} \sum_{k=0}^{\infty}{}' |V_k^N(f)| &\leqslant |V_0^N(f)| \sum_{k=0}^{\infty}{}' \kappa_*^{-k} \\ &= \frac{|V_0^N(f)| (1 + \kappa_*)}{2(\kappa_* - 1)} \\ &= O\left( |b_N^N| \frac{\kappa_*}{2(\kappa_* - 1)^2} \right). \end{aligned} \tag{5.13}$$

From (5.4) and (5.5), we also find that

$$|b_N^N| = O(\kappa_*^{-N}).$$

Thus the inequality (5.13) gives

$$\sum_{k=0}^{\infty}{}' |V_k^N(f)| = O\left(\frac{\kappa_*^{-N+1}}{2(\kappa_* - 1)^2}\right). \tag{5.14}$$

Summarizing the series of the above inequalities (5.11)–(5.14), we then get the following main theorem for this section.

THEOREM 5.1. *Let $f(z)$ is a meromorphic function which has M simple poles at the points $z_m$ ($m = 1, 2, ..., M$) outside of $\varepsilon_\tau$ with residues Resf($z_m$). Then for $f \in A(\varepsilon_\tau)$,*

$$|E_N f| = O(|b_N^N| \, H(\alpha; N))$$
$$= O\left(H(\alpha; N) \frac{\kappa_*^{-N+1}}{2(\kappa_* - 1)^2}\right), \tag{5.15}$$

*where $\kappa_* = \min_{1 \leqslant m \leqslant M} |z_m + \sqrt{z_m^2 - 1}| > 1$, $b_N^N$ and $H(\alpha; N)$ are given in (1.6) and (5.12), respectively.*

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